Chapter 28
History of Tools and Technologies in Mathematics Education

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1 Introduction

Since the advent of the electronic calculator, it has become customary for discussion of "technology" in mathematics education to refer almost exclusively to use of electronic devices. For example, in a daily newspaper in 2011, we find the following:

> Ever since the first elementary school teacher rolled the first television set into the first classroom to air the first course offerings from "educational television," there's been the hope and the promise that technology would revolutionize the way teaching and learning would be done. (Pearlstein 2011)

The implication here is that there was no such thing as technology in education before electronic technology and that there were no great hopes for revolutionizing teaching and learning prior to such technology. However, this narrow view is highly misleading. The employment of tools to assist teaching and learning of mathematics in fact has a history long predating electronic technology, and some of them have been proclaimed as revolutionary. In this chapter we will endeavor to look at the history of educational technology in a more integrated fashion, giving no special preference to electronic technology. Indeed, such an approach provides a useful perspective from which to view the debates surrounding the electronic tools of today.

In order not to go to the other extreme, with the concept of technology encompassing an unmanageably large range of human activities, perhaps including mathematical notation and language in general, we will limit ourselves to material devices. Thus, for example, we will not count logarithms as a technology, while the slide rule, a physical device based on logarithms, will be within our purview.

It must also be acknowledged that even within these bounds, this chapter fails to cover the history of technology in mathematics education uniformly across the globe. Space limitations, combined with the special interests of the present writer, have resulted in a treatment that often focuses on developments in the United States, occasionally provides brief discussions of education in Europe, and regrettably offers very little direct commentary on other parts of the world. The reader may also detect a presumption that there has been significant homogenization of educational technology worldwide in recent decades, with little effort to present any supporting evidence. It is hoped, nevertheless, that this chapter will be usefully provocative even for those with interests different from the writer and will suggest fruitful avenues for further research.

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We organize our discussion in relation to a technology's scope of use, classifying technology into two primary groups: general-purpose tools and specialized technologies. By general-purpose tools, we refer to those of wide importance in many walks of life outside classrooms but put to special use in an educational setting. The specialized technologies, in contrast, are most likely to be encountered in technical work such as science or engineering. Some of these have been explicitly developed for teaching mathematics and have been largely confined there.

Educational use of technology has been subject to overarching educational philosophies prevailing at any given time and place. We will comment on the influence of some of these philosophies (sometimes disparaged as fads or fashions) where appropriate.

2 General-Purpose Technologies Used in Mathematics Education

2.1 The Textbook

At times technology has been invented specifically to serve mathematical purposes. At other times technology has entered mathematics, and specifically mathematics education, from the larger world outside, notably from commerce and from science. Probably the most ubiquitous of such tools, retaining a powerful presence in worldwide mathematics education to the present day, is the book. As an educational tool, the book serves as a medium for storing and displaying information to be conveyed to students. The book has a history almost as old as civilization itself, from clay tablets to the papyrus scroll, to the handwritten codex, to the printed book, and on to the modern e-book (Hobart and Schiffman 1998; Schubring 1999, 2003.). The manifold contributions of this technology to civilization are well known and need not be recounted. But the history of the mathematics textbook is much shorter, especially if we neglect advanced monographs in favor of books actually used in schools. Certainly for many centuries, individuals have learned mathematics independently from books, and likewise tutors have used books to teach mathematics to individuals and small groups, but a new era begins with the advent of mass schooling and the mass-produced textbook. These interconnected phenomena did not become prominent until the nineteenth century in Europe and the Americas and were materially aided by both political and economic developments. On the political side, there was rising support for providing education for a larger proportion of children. On the economic side, there were increasing efficiencies in the production of the physical book, and increasing facilities for transporting them over long distances, resulting in the ability to manufacture and distribute large numbers of books relatively cheaply (Kidwell et al. 2008).

When books were scarce, if a school class had a book at all, it would frequently be the exclusive possession of the teacher. If the class was of any appreciable size, this encouraged the recitation method of teaching, which frequently entailed the teacher simply reading aloud from the book and the pupils attempting, through writing or brute memorization, to retain what was read and then to recite it back to the teacher. Notable attempts to scale this system up were made in England and its colonies in the late eighteenth and early nineteenth centuries with the so-called monitorial system, in which the teacher would first teach a group of more advanced students, who would in turn teach less advanced students. In mathematics in particular, the recitation method and the monitorial system primarily supported a curriculum centered on the rote learning of the rudiments of arithmetic (Butts 1966).

Prior to the emergence of both the textbook and the blackboard, it was also common practice in many schools in Europe and North America for each student to produce a "copybook" or "cyphering book." Beginning with a collection of blank pages (paper and binding quality could vary widely, depending on economic circumstances), the student would copy out the material spoken aloud by the teacher. In the case of a teacher reading from a printed book, this could often mean that the student was almost literally producing a handwritten copy of the book or the problems from the book. Here again the use of copybooks primarily supported arithmetic instruction, but in some cases this could be fairly elaborate, including square and cube roots and complicated problems from commerce and business. The teacher could periodically inspect the copybooks, so that they could have functioned as what more recent educators would term a "portfolio." But how rigorously eighteenth- and nineteenth-century copybooks were evaluated for mathematical correctness is unclear, and some may have been assessed more on aesthetic grounds, such as penmanship (Cohen 1982; Clements and Ellerton 2010).

But with cheaper books came the possibility (though still often not the reality) that students as well as teachers could have individual access to a textbook. A student with a book could now be asked to read that book both during and outside of class and to work problems assigned from the book. It was now easier than previously to provide more sophisticated mathematics instruction for a classroom of pupils. Thus, the rising presence of algebra and geometry in addition to arithmetic in the curriculum of nineteenth-century schools surely owes a good deal to the proliferation of textbooks. It is also likely that the use of textbooks served to hide problems with inadequate teacher preparation. This was certainly the case in the nineteenth-century United States (Tyack 1974).

Moreover, the system of textbook usage amplified itself: a greater supply of books produced a greater demand for books, which in turn produced yet more books, and so on. In mathematics this resulted not merely in the creation of individual textbooks but entire series of textbooks covering the whole range of the curriculum from the lowest grades to the colleges: basic arithmetic to the differential and integral calculus. Conditions in the United States, especially the free-market economy and the separation from Britain, seem to have been especially favorable for establishing a vibrant textbook industry in the nineteenth century. In the United States notable nineteenth-century authors of mathematics textbooks include Charles Davies, Joseph Ray, and George Wentworth (Kidwell et al. 2008). In contrast, Australia relied for far longer on British textbooks and was thus slower to establish its own textbook industry. The educational influence of Europe on colonized regions is complex and is the subject of recent scholarly attention (Ellerton and Clements 2008).

One notable effect of textbooks has been to standardize and codify curriculum. Educators have often found it difficult to dislodge curriculum topics once they are printed in widely distributed textbooks. This is especially striking in the United States, which despite a long tradition of local control of schools, and avoidance of an official national curriculum, rapidly converged on a de facto standard curriculum in mathematics, as a relatively small number of textbooks began to dominate the market. Genuinely innovative mathematics textbooks have never fared well in the US market. Even during the 1950s and 1960s, supposedly a time of major upheaval, we can observe important textbooks exhibiting substantial continuity from earlier decades. The largest American program for curriculum reform during that era, the School Mathematics Study Group (SMSG), produced a variety of text materials, which were published in an inexpensive format by the Yale University Press. The hope was that these texts, some highly innovative, would serve as models for commercial textbooks. But this hope was realized in only rare cases, the most successful of which was the Houghton Mifflin algebra textbook series with Mary Dolciani as the lead author. If one examines the Dolciani textbooks, it is clear that although there is a sprinkling of new material, they owe a great deal to Houghton Mifflin texts from the days prior to SMSG (Freilich et al. 1952; Dolciani et al. 1965; Wooton 1965; Roberts 2009).

2.2 The Blackboard

The blackboard or chalkboard and its offshoots are today widely used outside education, especially in business and government, but unlike the book this technology seems to have found its first extensive use in the classroom and only then moved outward. Educational use of this tool is tightly bound to the rise of mass education, which brought a pressing need for multiple individuals to view the same information simultaneously. Prior to the wall-mounted blackboard, there had been a slow evolution of handheld writing surfaces, culminating in the slate, which could be written on with chalk. In Europe and North America, this was often a facet of the recitation method of instruction. The teacher could read a problem from the book, and the students could copy and display their solutions on their slates (Cajori 1890; Burton 1850).

The erasable blackboard, written on with chalk, spread quietly into schools in the early 1800s and was well established by the end of that century (Kidwell et al. 2008). It allowed the teacher to display complicated verbal or pictorial details with far more exactitude than merely reading aloud from a book. Moreover, it allowed students to work out problems on the board themselves, displaying their efforts for both the teacher and for other students to see and comment on, thus changing the personal dynamics of the classroom. In mathematics the blackboard worked in conjunction with the textbook to promote the rise of both algebra and geometry in the curriculum.

Blackboards have continued in use in mathematics classrooms to the present time. In many cases, the chalkboard has been replaced by the "dry-erase" or "whiteboard," but with no essential change in functionality. The interactive whiteboard, developed in the late twentieth century, represents a major innovation, allowing the material displayed on the board to be connected directly to a computer. Opinions vary widely on the value of this technology in the classroom (Smith et al. 2005; Wood and Ashfield 2008). Tablet personal computers offer similar functionality, including handwriting recognition, whereby the computer is able to interpret handwriting drawn on the screen, not merely type entered via a keyboard (Anderson 2011).

2.3 The Overhead Projector

A more recent classroom display technology is the overhead projector. Its earliest manifestations seem to have been related to education but not in school classrooms: public nineteenth-century science lecturers seeking added visual flair. Such use began to enter schools in the early twentieth century, as part of a wider movement for "visual education" that included photographic slides and filmstrips. About the same time, this technology also received a boost from a noneducational venue, the bowling alley, where it was used as a convenient way to project scores for bowlers to view. Overhead projectors then received substantial use by the US military during World War II for training purposes, probably contributing to a major expansion of school use in the postwar years (Kidwell et al. 2008).

Much more than the blackboard, this technology as used in schools has remained the exclusive domain of the teacher. It has two primary attractions. First, it allows the teacher to continue to face the students while displaying materials to them. Second, it allows the teacher to display elaborate transparencies created before class. For example, a teacher of solid geometry can prepare or purchase complicated diagrams of an exactitude that could never be hoped for in hand-drawn diagrams quickly improvised while watched by the students. There is however a drawback, in that reliance on prepared slides can encourage a too rapid succession of material that can overload the students' ability to assimilate the information presented.

Overhead projectors have continued in use to the present but in many cases have been superseded by new technologies allowing greater ease of use and a greater range of display functionality. Computer projection systems permit the display of any image, static or moving, available to the host computer and in particular allow slide shows formerly done via transparencies on an overhead projector to be accomplished via software such as PowerPoint. Another enhancement of the overhead projector is the document camera (also known as an image presenter or visualizer), which permits any document, or even a three-dimensional object, to be displayed on the overhead screen without any prior preparation of the document or object (Ash 2009).

Many classrooms in the twenty-first century provide not only a computer and projector for the teacher but also a computer for each student, networked with the teacher's computer. In some ways this is a return of the handheld slate, with a vast increase in functionality. Its potential for mathematics instruction is just being tapped.

2.4 The Computer

Like the book, this tool's wider societal uses are enormous. It has now established itself in mathematics education throughout the world, although its ultimate role is perhaps not yet clear. It can be argued that much educational use of computers is trivial compared to the full capabilities of the technology. For example, many students today can read textbooks on a computer screen, but this is surely not a profound capability. Probably the most common use of computers in elementary instruction is to provide instant feedback to students working on problems. This is undoubtedly an increase in convenience that might amaze earlier generations of students and teachers, but in principle it is no different from looking up the answer in the back of the book.

Unlike the book, the advent of computers in education is not lost in the mist of time and indeed is still within living memory. The original "main-frame" computers, developed during and just after World War II, were too expensive, too bulky, and required too much maintenance to have much attraction for educators. It was only in the 1960s, with time-sharing systems and with so-called minicomputers that there began to be any appreciable use of computers in education. It was now possible for several students to simultaneously interact with the same computer. A pioneering instance was the University of Illinois's Project PLATO (Programmed Logic for Automatic Teaching Operations) (Bitzer et al. 1961). This was built upon earlier, nonelectronic, "programmed" learning efforts which had become popular beginning in the 1950s. Programmed learning experiments, much of which were inspired by the work of B. F. Skinner and other psychologists, featured ordered sets of problems which the student was asked to work through (Vargas and Vargas 1996). The student's passage through the problems depended on whether the student gave correct or incorrect answers at each step; a student might be asked to cycle back through some material or else move on briskly to new topics. This could be accomplished merely with a book, by covering up the answers. Computers allowed this to be done more easily and with more flexibility. As already noted, this basic functionality continues to be one of the most widely used applications of computers in mathematics education.

A different tactic for computer use in education was explored at Dartmouth College, again beginning in the 1960s. Here the aim was to have undergraduates program the computer themselves, thus learning the fundamental logical principles behind the machines. They succeeded in making computer programming a feature not only of mathematics classes but of other classes where mathematics was applied, including business and the social sciences. A key piece in achieving this was the development by Dartmouth professors John Kemeny and Thomas Kurtz of the BASIC computer language, which subsequently spread worldwide among novice and expert computer users alike. Ultimately one major effect of the Dartmouth work, and other similarly oriented projects throughout the world, was that computer science branched off from mathematics as a separate academic discipline at the college and university level (Kemeny and Kurtz 1985).

The emergence of the microcomputer, or personal computer, in the 1970s and 1980s, gave further impetus to educational use of computers, especially below the college level. For the first time computers became a home appliance, which made school use much more comfortable for both students and teachers. Computer games, some with an educational component, such as *Lemonade*, for the Apple II personal computer, began to proliferate (Apple 1982). And now that the crude teletype terminals of earlier days were being replaced by video display screens, it was possible to generate much more elaborate graphics, with obvious application in geometry instruction. *Geometer's Sketchpad* and

Cabri-géomètre are two examples of computer programs taking advantage of these capabilities (DeTurck 1993). Statistical software such as *Minitab* and algebra software such as *Derive* also came on the market in the 1980s (Ryan and Joiner 1973; Grinberg 1989). Large software packages incorporating a full range of algebraic capabilities, together with sophisticated graphics, included *Maple* and *Mathematica* (Chonacky and Winch 2005). Such software has raised as yet unanswered questions about the content and methods of mathematics instruction. Even general-purpose software such as Microsoft *Excel* offers extensive mathematical capability which potentially could totally reshape the mathematics curriculum.

It must be noted, however, that computer use in mathematics classrooms varies greatly worldwide. The cost of purchasing and maintaining computers, together with training instructors to use them effectively, remains a significant obstacle in many places, especially compared with the older technology, the book.

3 Specialized Technologies Used in Mathematics Education

In addition to general-purpose tools, mathematics education has made use of specialized tools. Some of these have originated outside education, especially in commerce, science, and engineering. Others have originated within education and then moved outside. A few are essentially unique to mathematics education. We classify them here into three broad categories: tools for calculation, tools for drawing and display, and tools for physical manipulation.

3.1 Calculating Tools

Calculation is an activity that many in the general public consider synonymous with mathematics, to the distress of many mathematicians and mathematics educators. Of course there can be little doubt that the historical roots of much mathematics are found in the practical need for calculation, and consequently calculation has been a central justification for mathematics education since antiquity. In general, physical tools for calculation have first received extensive use outside the classroom, in realms where speed and efficiency are more pressing issues, before becoming an accepted part of standard school instruction. The slide rule, for example, was a tool of practicing engineers for decades before it was seriously taught in schools. Possibly the abacus, as used in Asia, is an exception to this trajectory. No physical calculating device has been a part of mathematics instruction in the West in the manner, or for the long duration, that the abacus has been part of such instruction in Asia.

3.1.1 The Abacus

The abacus depicts numbers by means of beads on wires. It apparently evolved from marks in sand or counters on a board. The device seems to have developed somewhere in the eastern Mediterranean world in antiquity, moved east to Asia, then moved back west via Russia into Europe and thence to the Americas. The transmission to Asia is conjectural, and it is possible that it originated there independently. What is clear is that whereas the abacus became a widely used tool of calculation in China and Japan, without serious competitor until very recent times, it never attained the same level of popularity in this role in Europe and North America. Instead, in the latter regions, it was primarily confined to use as a demonstration tool for teaching elementary arithmetic to young children.

The Chinese abacus (*suanpan*) appears to have been in substantial use by 1200 and probably much earlier. Transmission to Japan seems to have occurred via Korea. The Japanese modification of this instrument (called the *soroban*) was in use by 1600 (Smith 1958). The abacus has been part of education in both nations for centuries, and the device has continued to be part of mathematics instruction in many East Asian nations to the present day, although not without some controversy and competition from newer technology. In Malaysia, for example, although abacus use in schools declined for a time after handheld calculators became widely available, the abacus (*sempoa* in Malay) has more recently experienced an educational resurgence in connection with an increased emphasis on mental arithmetic (*China Daily* 2010; Shibata 1994; Siang 2007).

While in East Asia the beads move on vertical wires, the version of the abacus that became common in Russia featured horizontal wires. This would prove advantageous for using it as a display device for young children, since the teacher could hold up the abacus in front of the class and the beads would remain in place. It was used in Russia for early education until recent decades. The French mathematician Jean Victor Poncelet encountered the abacus while imprisoned in Russia following Napoleon's invasion of 1812 and introduced it to France on his return. It spread widely across France as a teaching tool in the nineteenth century (Gouzévitch and Gouzévitch 1998; Régnier 2003).

A similar teaching device began to appear in the United States in the 1820s, likely inspired at least in part by the French version. Here it meshed well with the Pestalozzian object-teaching philosophy that was gaining in popularity, and by the 1830s, it was being sold under various names, including "numeral frame" by companies catering to the growing educational market. These teaching abaci were not without detractors, however, some of whom felt they might even stifle the imagination of the child. They remained as a tool for only the youngest learners of arithmetic (Kidwell et al. 2008). In more recent years, apparently reacting to the perceived success of Asian students in mathematics, some educators have advocated more use of the Asian abacus in Western schools (Ameis 2003).

3.1.2 The Slide Rule

The slide rule incorporates in physical form the theory of logarithms pioneered by Scottish mathematician John Napier and English mathematician Henry Briggs in the early 1600s. By marking two straightedges with logarithmic scales and sliding one with respect to the other, it was possible to quickly calculate approximate answers to multiplication problems. Even more complicated problems could be handled with sufficient ingenuity, although the fact that the slide rule was an analog instrument meant that it always provided only approximate answers and thus was not appropriate for accounting or other commercial applications. Variations involving circular rules were also possible, and both possibilities had been explored by the middle of the seventeenth century in England. These slide rules were slowly improved over the next century and became a tool used by British engineers such as James Watt. By the early 1800s, they had spread to the European continent and to the United States (von Jezierski 2000).

It was not until the late nineteenth century that the slide rule became an educational tool, beginning first with colleges featuring an engineering curriculum. In Britain, the engineer John Perry included the slide rule among the practical tools that he advocated for reforming the training of engineers, scientists, and mathematicians, which he promoted first at Finsbury Technical College and later at Imperial College, London (Perry 1913; Gooday 2004). In the United States, institutions such as Rensselaer Polytechnic, the United States Military Academy, and the Massachusetts Institute of Technology were leaders in educational use of this technology. In the early twentieth century, the slide rule began to filter down into the secondary schools, helped by the movement, in both Europe and the United States, to establish mathematical "laboratories" which emphasized the mathematics of measurement and applications to the physical sciences. Instrument makers were selling slide rules to the high school market by the 1920s and some were also selling oversized models that could be displayed in front of a classroom for all students to see. The slide rule remained a recognized feature, although in most cases not a central one, of many mathematics and science classrooms until the advent of cheap electronic calculators in the 1970s (Kidwell et al. 2008).

3.1.3 The Calculator

Unlike the slide rule, the calculator is fundamentally a digital instrument, which seems to have given it a decided advantage in achieving a place in mathematics instruction. Its place in the classroom is still in an experimental stage. European development of mechanical calculators dates from the seventeenth century, with such notable mathematicians as Pascal and Leibniz prominently involved (Goldstine 1972). But it was not until the middle of the nineteenth century that industrial processes were sufficiently advanced to allow construction of calculating devices on a commercial basis, both in Europe and the United States. By the 1920s they had become a standard feature of many office settings. But it appears that it was not until after World War II that they received much consideration as educational assistants. In the 1950s there was some minor experimentation in classrooms with mechanical calculators, or mechanical calculators with electrical assistance, but the size and cost of these machines made them inconvenient as personal devices (Kidwell et al. 2008).

The major breakthrough occurred in the 1970s, with the arrival of inexpensive, fully electronic calculators. Initially these calculators were still relatively bulky and were able to perform little beyond the familiar four operations of arithmetic. But by the 1980s calculators had become readily portable and were able to compute trigonometric and other transcendental functions and to display graphs, thus far surpassing the functionality of mechanical calculators and slide rules. Classroom use became practical and although very uneven, soon became widespread enough to create disputes between enthusiasts and detractors. Calculators greatly increased the range of feasible problems that could be given to students, but concern was expressed about the effect on basic arithmetic skills, and doubts were raised about the readiness of teachers to use calculators effectively (Kelly 2003; Waits and Demana 2000). By the mid-1990s computer algebra systems (CAS) were available on handheld devices, leading to further debate. Now, in the twenty-first century, although the generic name persists, high-end devices referred to as “calculators” in fact provide a huge range of information storage, information display, and demonstration capabilities, in addition to pure calculation (Aldon 2010; Trouche 2005). Some controversy has persisted, but in recent years the use of calculators has been increasing around the world in secondary and elementary schools and at the college level as well.

3.2 Tools for Drawing and Display

Among such tools we include both devices for making marks and the media upon which the marks are made. (That the slate and blackboard could be considered in this latter category shows that our classification scheme is far from clear cut.) Most of these tools have found abundant use outside of education, most especially by surveyors and engineers. The most ubiquitous of such tools, the straightedge and the compass, are so old that their origin in education or anywhere else is highly obscure. It is of course well known that the ancient Greeks sought to investigate which figures could be constructed with straightedge and compass alone and that this led to classrooms throughout the West featuring these instruments in geometry instruction. We can add very little to this general outline here. Instead we will focus on some more recently invented devices, used less universally, but whose histories are nevertheless revealing.

3.2.1 The Protractor

The protractor, a semicircular device with markings to measure degrees of angles, emerged in Europe in the sixteenth century, out of the confluence of tasks generated by surveyors, navigators, and mapmakers. Early protractors were generally made from brass or horn. Horn, though less rugged than brass, and subject to wrinkling or curling, offered the advantage of being semitransparent, so that a draftsman could see an existing drawing underneath the protractor. By the eighteenth century, discussion of protractors began to appear not only in manuals for instrument makers but also in geometry textbooks, especially in France (Kidwell et al. 2008).

Geometry instruction remained more formal in the English-speaking world well into the nineteenth century. In those texts where straightedge and compass constructions were emphasized, protractors were not considered appropriate. A few "practical geometry" textbooks began to appear in the early nineteenth century, but it was not until the middle of the century that there was substantial movement away from formal Euclidean treatments of geometry. In the United States this approach, much later dubbed "informal geometry," was driven in part by educational philosophies emphasizing greater emphasis on using sense data, especially visual, to convey the abstract concepts of mathematics. The Swiss educator Johann Pestalozzi and his follower Friedrich Froebel were influential in this regard. American reform educators also observed the contemporary German efforts to develop geometry instruction for those not intending to attend university (Coleman 1942).

American geometry instruction through the remainder of the nineteenth century featured an eclectic mixture of formal and informal and of varying focus on the practical utility of geometry versus the merits of the subject for training the mind. Harvard president Thomas Hill's geometry textbook of 1863 explicitly directed protractor use for solving many of its problems and even described how students could create their own instruments (Hill 1863). But other books, such as those of Charles Davies and George Wentworth appearing later, continued to make little concession to practical matters, treating geometry as a purely abstract subject in which the protractor had no place (Davies 1885; Wentworth 1877). A rapprochement began to be effected in the 1890s as part of the general effort to standardize the entire secondary school curriculum. The formula adopted, first enunciated by the mathematics subcommittee of the Committee on Secondary School Reform (better known as the "Committee of Ten"), was to urge initial geometry instruction to be "concrete," while older students would be taught in a rigorously "demonstrative" manner. The protractor was an important tool for the former but was laid aside for the latter. Soon after this period, instrument manufacturers began to market much cheaper protractors (of cardboard or celluloid) for the growing demand. The protractor has continued to hold a similar place in geometry instruction to the present (Kidwell et al. 2008).

3.2.2 Linkages

Although the straightedge was long considered an unproblematic instrument, there was a brief period in the nineteenth century where there was agitation to change this. The impetus originated in engineering, specifically with the work of James Watt. In the course of refining his steam engine in 1784, he devised a system of rods and pins to convert rotary motion to approximately straight-line motion. This later caught the attention of Russian mathematician P. L. Chebyshev, who asked a question that seemingly had never before been asked explicitly: is it possible to produce an exact straight line by mechanical means? Whereas the compass provides a means of producing an exact circle with the simplest of means, it is not at all obvious how to produce a straight line in a similar fashion, without simply tracing along an already-existing straight line, which is how a straightedge is conventionally used (Kidwell et al. 2008).

The problem was solved in 1860s and 1870s by use of inversive geometry. A point on a system of rods or bars, connected by hinges or pivots, could be made to trace an exact straight line as another point on the device was made to traverse a circle. The discovery of these devices produced a brief flurry of intense interest among some mathematicians. English mathematician J. J. Sylvester invented the term "linkage" to describe all such systems of rods and pins (Hilsenrath 1937). It was shown how to produce other curves and to perform such feats as trisecting angles (Yates 1945). There were even calls to refashion geometry education. In 1895, the American mathematician G. B. Halsted unsuccessfully proposed the following:

> Henceforth Peaucellier's Cell and Hart's Contraparallelogram [two linkages producing exact straight lines] will take their place in our text-books of geometry, and straight lines can be drawn without begging the question by assuming first a straight edge or ruler as does Euclid. (Halsted 1895)

These devices have never become more than an enrichment topic in the classroom (Kidwell et al. 2008), but they have continued to create enthusiasm among mathematics teachers and teacher educators to the present time (Bartolini Bussi and Maschietto 2008).

3.2.3 Graph Paper

Graph paper, a now familiar medium for depicting geometric figures, has a shorter history than is often realized. Although today's student of "Cartesian" geometry is often requested to "graph the following equation," such problems are foreign to Descartes' own seventeenth-century work. Indeed, it was only in the nineteenth century that this procedure became a standard part of the mathematical repertoire and not until the twentieth century that it became entrenched in school instruction.

Special ruled paper, designed to facilitate the depiction of relationships between two varying quantities, is essentially a nineteenth-century innovation of civil engineers, although some intimations can be found in eighteenth-century astronomy and chemistry. Builders of roads, canals, and especially railroads found it increasingly important to compare the vertical change on a route in relation to its horizontal progress. At first, individual users created their own paper to accomplish such tasks, but by the 1870s commercially produced paper was available. The cost of such paper rapidly declined in the last decades of the nineteenth century, making possible its use as an educational tool in the twentieth (Kidwell et al. 2008).

But engineering use and cheap production costs would not in themselves have created demand for graph paper in mathematics instruction, were it not for changes in the philosophy of mathematics education. The case is especially clear in Britain and the United States, both of which countries experienced reform movements related to emphasizing the value of visualizing abstract concepts. In Britain, John Perry, already mentioned, promoted a more concrete and visual approach to mathematics education, helping to break the unquestioned dominance of formal Euclidean geometry in British education. His influence extended to both Japan (where he worked for a time in the 1870s) and the United States. One of his specific proposals was for the substantial use of "squared paper" to facilitate mathematics instruction at all levels (Brock 1975; Brock and Price 1980). In the United States, Perry's most prominent disciple was pure mathematician E. H. Moore of the University of Chicago, who championed a "laboratory method" of teaching mathematics at both the secondary and college levels. This involved strong emphasis on developing intuition in the student through physical models, weighing and measuring, and drawing on squared paper. Moore hoped to help students aiming to be scientists and engineers while also supporting future teachers of mathematics and research mathematicians. Moore's long-term influence on the American curriculum was slight, one major exception being an increased use of graphs in algebra instruction, something rarely found in the nineteenth century (Roberts 2001).

Between 1910 and 1930, graph paper became established as a regular feature of much mathematics instruction in the United States. The picture-free algebra books of the nineteenth century were replaced by a new generation of textbooks containing pictures of graphs on grids of perpendicular lines and featuring problems inviting students to graph equations and other mathematical objects. Graphing has continued to have a regular place in mathematics instruction to the present time, although the role of graph paper itself is often replaced by the graphing calculator (Kidwell et al. 2008).

3.3 Tools for Physical Manipulation

In this category we consider any device that primarily serves its purpose by being physically handled and examined in three dimensions. In Europe and North America, there has been a discernable increased use of such tools from the beginning of the nineteenth century, although even within this period, the history is often strikingly erratic (Bartolini Bussi et al. 2010). Pestalozzi and Froebel, already mentioned, were especially influential in bringing material objects into the classroom to be seen or touched by the students. These included objects associated with mathematics, such as geometric solids. Froebel, teaching in Swiss and German towns in the 1830s and 1840s, recommended organized play with blocks to introduce the child to geometric shapes and to arithmetic ideas up to simple fractions. Froebel's ideas spread across Europe and to the United States in the late nineteenth century (Allen 1988).

An example of a tool of this kind that has come and gone with little trace is the cube root block. It is based on a method of extracting cube roots based on the binomial expansion of $(a+b)^3$, which can be illustrated with a cube of side $a+b$. (There is a corresponding method for extracting square roots, more well known, which can be illustrated with a diagram of a square of side $a+b$.) Illustrations of this cube can be found in English arithmetic texts from the seventeenth century (Recorde 1632), but it was not until the middle of the nineteenth century that it became an actual classroom device. With the aim of helping students understand the aforementioned cube root algorithm, scientific instrument companies in the United States began to produce and market wooden cube root blocks that could be dissected into constituent parts. These blocks, for advanced arithmetic students, were often advertized in conjunction with other classroom objects, such as cones for displaying conic sections and Froebel's blocks for kindergarten children. Diagrams based on the blocks were a staple of school arithmetic textbooks for many years, but the topic had detractors. The cube root block algorithm never gained any favor with engineers and other users of mathematics for practical purposes, since the efficiency of the algorithm is low compared to other methods, such as logarithms or Newton's method. Moreover, how often did mathematical practitioners even need to compute cube roots? By the 1890s many mathematics educators in the United States were campaigning against cube root extraction, but it persisted in the curriculum well into the twentieth century. Cube root blocks were still being sold in the 1920s. No studies of the effectiveness of the cube root block as a teaching technique are known. The block must be judged a demonstration tool of unclear benefit to support an algorithm of dubious value, but nevertheless for a time, it was a standard topic in the schools (Kidwell et al. 2008).

The cube root block can also be considered as part of a wider movement in Europe and North America to use geometric models in classrooms. This is built on a tradition originating in France in the early nineteenth century, especially with mathematician Gaspard Monge. Models made of plaster, string, wood, metal, and paper were developed in France and Germany. These went beyond the simple solids of Pestalozzi and Froebel to include hyperboloids and other more advanced structures, all the way to objects at the forefront of mathematical research, such as Riemann surfaces. Some of the string models could be manipulated to change shape. In Germany in the 1880s, at the instigation of the prominent mathematician Felix Klein, models, mainly of plaster, were manufactured and sold worldwide. Colleges and universities in the United States were among the buyers, but there is little evidence to support extensive classroom use of these models; more likely they were treated more as museum pieces. There were also isolated enthusiasts in the United States in the twentieth century, who built models or helped students build models. Their influence is very hard to gauge (Kidwell et al. 2008). In France, in this period, mathematicians Émile Borel and Jules Tannery encouraged construction of models by both teachers and students as part of their "laboratoire d'enseignement mathématique" (Châtelet 1909).

Meanwhile in Italy, Maria Montessori inherited Froebel's emphasis on teaching young children through tactile experience, buttressing her theories by appealing to more recent developments in psychology and anthropology. She advised that beginning students be given the opportunity to continually handle objects of various shapes, such as cylinders of varying heights and diameters. Colored cubes and rods were a central feature of her approach to arithmetic. Montessori schools were opened in Italy and Switzerland. After an initially rapid growth of interest in her work in the United States in the 1910s, her influence declined, in part due to criticism from American educational theorists such as William Heard Kilpatrick of Columbia University (Whitescarver and Cossentino 2008).

The United States experienced a Montessori revival beginning in the 1950s, and this closely coincided with, and perhaps helped to support, renewed interest in both the United States and Europe in using physical objects specifically in teaching mathematics. Other sources of support were found in the work of educational psychologists whose influence extended well beyond mathematics, such as the Swiss, Jean Piaget, and the Russian, L. S. Vygotsky. Among those in the 1960s who helped popularize what came to be called "manipulatives" in mathematics instruction were the Belgian educator Emile-Georges Cuisenaire, the Egyptian-born British educator Caleb Gattegno, and the Hungarian-born educator Zoltan Dienes, who worked in Britain, Australia, Canada, and elsewhere (Jeronnez 1976; Seymour and Davidson 2003). This period also saw a ferment of curriculum reform, notably in France, the U.S.S.R., and the United States, but present to varying degrees in many other nations. Some would see manipulatives such as Cuisenaire rods as incongruous with the emphasis on axiomatics and abstraction characteristic of many of the "New Math" programs (to use the designation popular in the United States), although Dienes, for one, saw no contradiction (Dienes 1971). It does appear that the popularity of certain manipulatives to some extent rose and fell with public perceptions of the New Math as a whole. Nevertheless, while New Math programs often experienced severe backlash, the use of manipulatives never went into total eclipse.

The presence of manipulatives in classrooms in the last 50 years is reflected in the large quantity of empirical research on the topic from the 1960s to the present. This research paints a mixed picture of the effectiveness of these tools. While some studies have detected very positive effects, others find these effects negated by poor teaching techniques (Karshmer and Farsi 2008; Moyer 2001; Sowell 1989). Some research even suggests that manipulatives can harm students by burdening them with the problem of "dual representation."

> That is, a given manipulative needs to be represented not only as an object in its own right but also as a symbol of a mathematical concept or procedure (McNeil and Jarvin 2007).

The computer, especially as connected to the Internet, makes readily available to students and teachers all of the objects mentioned above, and many more, in virtual form. Is this comprehensive technology platform something fundamentally new for mathematics education, or does it merely provide the means for delivering the services of the older technologies more quickly and efficiently? It remains to be seen.

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